Constraints on brane-localized gravity

Philip D. Mannheim*

Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 11 September 2000; published 27 December 2000)

In this paper we explore some general aspects of the embeddings associated with brane-localized gravity. In particular we show that the consistency of such embeddings can require (or impose) very specific relations between all the involved bulk and brane matter source parameters. We specifically explore the embeddings of 3-branes with non-zero spatial 3-curvature k into 5-dimensional spacetime bulks, and show that for such embeddings, a 5-dimensional bulk cosmological constant is not able to produce the exponential suppression of the geometry thought necessary to localize gravity to the brane.

DOI: 10.1103/PhysRevD.63.024018 PACS number(s): 04.50.+h

I. GENERAL INTRODUCTORY REMARKS

It has been suggested recently [1–4] that it is possible for our 4-dimensional universe to be a 3-brane[1] embedded in some higher dimensional bulk spacetime whose spacelike extra dimensions need not in fact be as minuscule as their string theory Planck length expectation. And while the original motivation of these studies was an attempt to solve the hierarchy problem, nonetheless the potential existence of any large such extra dimension is a matter of great interest in and of itself.[2] Moreover, Randall and Sundrum [3,4] were able to show that the embedding of a flat Minkowski 3-brane into a 5-dimensional anti–de Sitter (AdS_5) spacetime would then explicitly localize gravity to the 4-dimensional world, thereby releasing the extra higher dimension from needing to be tiny. Now, while this is a very nice property of the AdS_5 embedding, it is important to see just how generic it in fact is and to what extent the embedding into a 5-dimensional spacetime of any given 4-dimensional set of matter fields in any given 4-dimensional geometry[3] would in fact then result in a gravity that actually was localized to the 4-dimensional world. Moreover, it is equally necessary to see whether any given such 4-space configuration of matter fields can even be consistently embedded into a higher dimensional space at all. And, indeed, in one sense the whole localization issue is initially somewhat puzzling, since gravitating material always produces a gravitational field in the empty space around it, and it is straightforward to produce source configurations in which the associated gravitational potentials can actually grow at distances far from such sources. While an immediately obvious example of such a source might be a uniform density, zero curvature sheet of non-relativistic static gravitating material, a source which produces a constant, non-declining, Newtonian gravitational force away from the sheet when the sheet is immersed in an otherwise flat, empty background, such a non-relativistic gravitational field is just a coordinate artifact, being equivalent to a uniform acceleration in flat space. Thus, as we will show below, an embedding of such a sheet into a bulk with a non-zero 5-dimensional cosmological constant Λ_5 does in fact produce a gravity which is localized to the sheet, though, as we shall also see, the covariantizing of such a sheet (to then produce a true gravitational field) proves to be instructive, with the consistency of its embedding (even into a source free bulk) being found to only be achievable for very specific brane equations of state. Motivated by this analysis, we shall then extend our study to the case where the static sheet or brane is endowed with a non-zero spatial 3-curvature k (a configuration whose embedding into a source free bulk leads to a gravitational force which even grows with distance, one which is not a coordinate artifact), to then find in this case that the bulk Λ_5 is found incapable of producing exponentially suppressed localization of the geometry to the brane.

To begin our analysis it is instructive to recall some general properties of AdS_5 spacetime itself. As well as being constructible as a constant surface in a flat 6-dimensional space, the AdS_5 metric can also be given by the convenient form

$$ds^2=(R^2/z^2)(dz^2-dt^2+d\overline{x^2}) \tag{1}$$

where R is the radius of curvature. To see that this metric is in fact an AdS_5 metric, we note that since this metric is conformal to flat, we can explicitly determine its associated curvature by conformally transforming the flat $\eta_{\mu\nu}(x)$ met-

*Permanent address: Department of Physics, University of Connecticut, Storrs, CT 06269. Email address: mannheim@uconnvm.uconn.edu

[1]The dimensionality of a brane is defined by the number of its spatial dimensions just like that of a sheet of material.

[2]For a recent compilation of some of the rapidly growing literature in this field see e.g. [5].

[3]In discussing embeddings it is important to recognize that the geometry of a given bulk can be modified by the introduction into the bulk spacetime of a brane with its own specific symmetry, and with the brane setting up its own gravitational field in the bulk, it is important to ascertain whether a given AdS_5 bulk geometry remains so after the embedding of the brane. To illustrate the point we recall that the familiar Schwarzschild de Sitter metric $ds^2=-B(r)dt^2+dr^2/B(r)+r^2d\Omega$ [where $B(r)=1-2MG/r-kr^2$] is a metric for which the geometry exterior to the source is not maximally 4-symmetric $[R_{2323}=-kr^4(1+2MG/kr^3)\sin^2\theta\neq-k(g_{22}g_{33}-g_{23}^2)]$, with it only becoming so asymptotically far from the source. In the presence of the mass source then the geometry exterior to the source is not the dS_4 geometry it would have been had the source not been there.

ric according to $\eta_{\mu\nu}(x)\rightarrow\Omega^2(z)\,\eta_{\mu\nu}(x)=g_{\mu\nu}(x)$ where $\Omega(z)=R/z$. Under such a transformation the initially zero 5-dimensional Ricci tensor is found to transform to

$$R^{\mu}{}_{\nu}\rightarrow\Omega^{-5}\partial_\rho\partial^\rho(\Omega^3)\,\delta^\mu_\nu/3-3\Omega^{-1}\partial^\mu\partial_\nu(\Omega^{-1})=4\,\delta^\mu_\nu/R^2. \tag{2}$$

Moreover, since the Weyl tensor vanishes in geometries which are conformal to flat, the Riemann tensor associated with the metric of Eq. (1) is determinable from its associated Ricci tensor $R_{\mu\nu}=4g_{\mu\nu}/R^2$ alone, with it then immediately being found to take the form $R_{\lambda\rho\sigma\nu}=-(g_{\sigma\rho}g_{\lambda\nu}-g_{\nu\rho}g_{\lambda\sigma})/R^2$. We thus recognize the spacetime associated with the metric of Eq. (1) to be that of a 5-space with constant negative curvature $K=-1/R^2$, viz. AdS_5. As the above analysis shows, we could construct metrics of the form $ds^2=(R^2/x^2)(dz^2-dt^2+d\,\overline{x^2})$ where x is any one of the four spacelike coordinates and still have an AdS_5 spacetime. However, because of the signature change in the flat d'Alembertian operator $\partial_\rho\partial^\rho$, the metric $ds^2=(R^2/t^2)(dz^2-dt^2+d\overline{x}^2)$ would have constant positive curvature $K=+1/R^2$ and thus be a de Sitter rather than an anti–de Sitter space. For negative curvature spaces then the multiplying overall factor in the metric of Eq. (1) must only be associated with one of the spacelike coordinates. Since for such coordinates the transformation $z=Re^{y/R}$ allows us to rewrite the metric in the form

$$ds^2=dy^2-e^{-2y/R}(dt^2-d\overline{x^2}), \tag{3}$$

we see that in AdS spaces the spatial exponential $e^{-2y/R}$ factor acts just like its temporal analog $e^{2t/R}$ in de Sitter spacetimes. Such exponential behavior in AdS spaces is thus the spatial analogue of inflation, with the $e^{-2y/R}$ factor leading to rapid suppression as we go out in y away from the 4-dimensional space associated with the metric $dt^2-d\overline{x}^2$. Now since the multiplying factor in the metric of Eq. (1) is quadratic in z (and uniquely so for AdS spaces[4]), we can make transformations of either of the form $z=\mathrm{Re}^{+y/R}$ and $z=\mathrm{Re}^{-y/R}$ on the metric of Eq. (1). Thus we can make the transformation $z=\mathrm{Re}^{+y/R}$ in the $y>0$ region and the transformation $z=\mathrm{Re}^{-y/R}$ in the $y<0$ region to then give us $e^{-2|y|/R}$ exponential suppression for every value of y, positive or negative. However, now the two regions in y will be two separate patches of AdS_5 with there then necessarily being a discontinuity at $y=0$ where the two patches meet. It is thus at just such a discontinuity that our 4-dimensional universe can be located with our universe then being a 3-brane embedded in a higher dimensional bulk space containing two separate geometrical patches, with gravity potentially then being localized to the brane through the $e^{-2|y|/R}$ suppression [3,4].[5] It is thus to the implications of the embedding of such brane universes, and to the dynamical interplay of the bulk and the brane entailed by the very fact of such embeddings (even when the bulk itself contains no matter fields at all), to which we now turn, first in a source free empty background and then in one with a bulk Λ_5 (viz. a bulk that would be AdS_5 in the absence of the brane).

[4]Multiplying a flat space metric by any conformal factor will always lead to a new metric which is conformal to flat. However, it is only the choice $\Omega^2(z)=R^2/z^2$ which leads to a spacetime with the same maximal number of Killing vectors as the original flat spacetime itself.

II. EMBEDDING A BRANE IN AN EMPTY BULK

For the embedding of a homogeneous, isotropic standard 4-dimensional Robertson-Walker universe with spatial 3-curvature k into an arbitrary (and thus not necessarily AdS_5) 5-dimensional bulk space, the most general allowed maximally r,θ,ϕ 3-symmetric metric takes the generic form

$$ds^2=-n^2(y,t)dt^2+a^2(y,t)[dr^2/(1-kr^2)+r^2d\Omega]+b^2(y,t)dy^2+2c(y,t)dtdy, \tag{4}$$

up to arbitrary coordinate transformations involving y and t. Recognizing the pure y,t sector of this metric to be the most general 2-dimensional metric in a y,t space, we can thus make a coordinate transformation in this space to remove the y,t cross term, and in the illustrative static limit which we study here can then reabsorb $b(y)$ into a redefinition of y to thus yield

$$ds^2=-n^2(y)dt^2+a^2(y)[dr^2/(1-kr^2)+r^2d\Omega]+dy^2 \tag{5}$$

a metric whose embedding and localization of gravity aspects we now study.[6] While in brane-localized studies it is desired to recover standard gravity only in the 4-dimensional world, it is conventional to assume that the full 5-space gravity is given simply by the 5-dimensional Einstein equations (rather than by some more complicated set of 5-dimensional equations), viz.

$$G_{AB}=R_{AB}-g_{AB}R^C{}_C/2=-\kappa_5^2[T_{AB}+T_{\mu\nu}\delta^\mu_A\delta^\nu_B\delta(y)] \tag{6}$$

where T_{AB} ($A,B=0,1,2,3,5$) is due to sources in the bulk and $T_{\mu\nu}$ ($\mu,\nu=0,1,2,3$) is due to sources on the $y=0$ brane. For the symmetry of Eq. (5) both of these energy-momentum tensors are given as perfect fluids, viz.

$$T^A{}_B=\mathrm{diag}(-\rho_B,P_B,P_B,P_B,P_T),$$
$$T^\mu{}_\nu=\mathrm{diag}(-\rho_b,p_b,p_b,p_b) \tag{7}$$

(B denotes bulk and b denotes brane). For the metric of Eq. (5) it is straightforward to write the 5-dimensional Einstein

[5]The dependence of the geometry away from the brane on $|y|$ rather than on y itself is characteristic of the requirement made in all brane localized gravity studies (this one included) that there is to be a $y\rightarrow-y$ symmetry of the metric around the $y=0$ brane.

[6]While we concentrate here on embedding and localization issues, the brane theory associated with Eq. (4) has also been studied as a cosmology in and of itself; see e.g. [6–8].

equations, with the resulting expressions simplifying to (the prime denotes differentiation with respect to y)

$$G_{00}=3e^2f''/2f^2-3e^2k/f^2=-\kappa_5^2e^2[\rho_B+\rho_b\delta(y)]/f \quad (8)$$

$$G_{ij}=[-f''/2-fe''/e+k]\gamma_{ij}=-\kappa_5^2f[p_B+p_b\delta(y)]\gamma_{ij} \quad (9)$$

$$G_{55}=-3f'e'/2fe+3k/f=-\kappa_5^2P_T \quad (10)$$

when the identification $a(y)=f^{1/2}(y)$, $n(y)=e(y)/f^{1/2}(y)$ is made [here $\gamma_{ij}=\mathrm{diag}(1/(1-kr^2),r^2,r^2\sin^2\theta)$].

While we shall discuss the implications of these equations in various situations below, we note immediately that when the bulk T_{AB} is set to zero and when the spatial i,j coordinates are restricted to a flat 2-dimensional (x,y) plane and y is replaced by the usual spatial z, Eq. (5) then describes a uniform, infinite, flat 2-dimensional sheet of static matter embedded in (what otherwise would have been) ordinary empty spacetime, a system whose Newtonian limit is known to correspond to a gravitational potential which grows linearly with z and a gravitational force $F(z)$ per unit mass which is independent of z. Moreover, in such a case, the Newtonian gravitational force points toward the $z=0$ sheet no matter which side we consider, with Gauss' law yielding $F(z=0^+)-F(z=0^-)=4\pi G\sigma$ and $F(z=0^+)=-F(z=0^-)=2\pi G\sigma$ for a sheet of surface matter density σ, with the gravitational potential $\phi=2\pi G\sigma|z|$ thus being discontinuous across the surface.[7] As such the relation $F(z=0^+)-F(z=0^-)=4\pi G\sigma$ is a non-relativistic analogue of the fully covariant relativistic Israel junction conditions [9] (see e.g. [10,11] for some recent derivations)

$$K_{\mu\nu}(y=0^+)-K_{\mu\nu}(y=0^-)=-\kappa_5^2(T_{\mu\nu}-q_{\mu\nu}T^\alpha{}_\alpha/3) \quad (11)$$

across a discontinuous surface with normal n^A where $q_{AB}=g_{AB}-n_An_B\equiv q_{\mu\nu}$ is the induced metric on the surface and $K_{\mu\nu}=q^\alpha{}_\mu q^\beta{}_\nu n_{\beta;\alpha}$ is its extrinsic curvature. We shall thus expect to see an analogue of this Newtonian gravity discontinuity in the treatment of the relativistic case associated with Eq. (5), something to which we now turn.

For the simplest case first of a $k=0$ spatially flat Robertson-Walker geometry embedded in a source free bulk, on taking the metric coefficient $a(y)$ to be a function of $|y|=y[\theta(y)-\theta(-y)]$ [where $|y|'=\theta(y)-\theta(-y)$, $|y|'^2=1$, $|y|''=2\delta(y)$], integration of Eq. (8) is then found to yield

$$a^2(y)=\alpha(1-\kappa_5^2\rho_b|y|/3) \quad (12)$$

where α is an arbitrary constant which can be absorbed in a redefinition of the spatial x_i coordinates. With Eq. (10) then obliging $e(y)=e(|y|)$ to be a constant, we can then set $n(y)=1/a(y)$, with Eq. (9) then recovering the solution of Eq. (12) provided

$$p_b=-\rho_b/3. \quad (13)$$

As a check of this solution, we note that the Israel junction conditions which follow from Eq. (11) in our particular case, viz. [7]

$$[a'(y=0^+)-a'(y=0^-)]/a(y=0)=-\kappa_5^2\rho_b/3, \quad (14)$$

$$[n'(y=0^+)-n'(y=0^-)]/n(y=0)=\kappa_5^2(3p_b+2\rho_b)/3, \quad (15)$$

are indeed satisfied by our obtained discontinuity.[8] With $n^2(y)\to 1+\kappa_5^2\rho_b|y|/3$ in the weak gravity limit, we see that we nicely recover the linear potential characteristic of a Newtonian gravity sheet (though, as will be clarified below, one actually not with the standard weak gravity coefficient), with gravity not at all being localized to the brane and with the strong gravity limit even possessing a singularity at $|y|=3/\kappa_5^2\rho_b$.

While we thus see that we can obtain the anticipated nonlocalized solution, we find that it is only obtainable for a very particular equation of state, one with a negative pressure.[9] In order to understand this result we need to distinguish between the role that gravity plays in an ordinary 4-dimensional world and the one that it appears to be playing in the embedded case. As regards first the conventional pure 4-dimensional situation with no embedding into a fifth dimension, we note that there the fluid equation of state is usually taken as a fixed, gravity independent input, and the

[7]With brane studies always requiring the gravitational potential to only be a function of $|z|$, we see that such a requirement nicely dovetails with the constraints of Gauss' law.

[8]With all of the metric coefficients being functions of $|y|$, the junction conditions require the terms linear in $|y|$ to be first order in κ_5^2, viz. $a(|y|)=\alpha(1-\kappa_5^2\rho_b|y|/6)+O(|y|^2)$, $n(|y|)=\beta[1+\kappa_5^2(3p_b+2\rho_b)|y|/6]+O(|y|^2)=\beta(1+\kappa_5^2\rho_b|y|/6)+O(|y|^2)$. The junction conditions thus generate a contribution to the bulk Riemann tensor [see Eq. (41) below] which only begins in order κ_5^4, something which is to be expected since the order κ_5^2 constant Newtonian gravitational acceleration associated with a metric with $n(y)=\beta(1+\kappa_5^2\rho_b|y|/6)$ is removable by a coordinate transformation, with coordinate independent gravitational effects thus only beginning in order κ_5^4, with true gravity thus needing the $O(\kappa_5^2)$ terms of both of the $n(y)$ and $a(y)$ metric coefficients to be nonzero.

[9]To see that this is in fact a generic effect we note that for a flat 2-brane embedded in an empty 4 space, viz. one described by the metric $ds^2=-n^2(z)dt^2+a^2(z)(dx^2+dy^2)+dz^2$, the 4-dimensional Einstein equations $G_{\mu\nu}=-\kappa_4^2T_{\mu\nu}$ take the form $G^0{}_0=-(2aa''+a'^2)/a^2=\kappa_4^2\rho_b\delta(z)$, $G^x{}_x=-(a''n+a'n'+an'')/an=-\kappa_4^2p_b\delta(z)$, $G^z{}_z=-a'(a'n+2an')/a^2n=0$, with solution $a(z)=1/n^2(z)=(1-3\kappa_4^2\rho_b|z|/8)^{2/3}$, constraint $p_b=-\rho_b/4$, Israel junction conditions of the form $K_{\mu\nu}(y=0^+)-K_{\mu\nu}(y=0^-)=-\kappa_4^2(T_{\mu\nu}-q_{\mu\nu}T^\alpha{}_\alpha/2)$ (viz. $[a'(y=0^+)-a'(y=0^-)]/a(y=0)=-\kappa_4^2\rho_b/2$, $[n'(y=0^+)-n'(y=0^-)]/n(y=0)=\kappa_4^2(2p_b+\rho_b)/2$), and a Riemann tensor which is again of order κ_4^4. In passing we note also that our result confirms an old result of Vilenkin [12] that no static solution is possible for 2-branes with equation of state $p_b=-\rho_b$. However, we now see that a static solution is possible when $p_b=-\rho_b/4$.

gravity which it produces is then determined as output. Nonetheless, even in that case the p_b/ρ_b ratio need not necessarily be positive. Thus even while the energy density and pressure of a high temperature ideal gas due to the kinematic motions of the gas particles are both positive, if the gas is cooled into a solid phase, it then undergoes a phase transition, a long range order effect such as condensation into an ordered crystal lattice, an effect which can be associated with the negative pressure characteristic of vacuum breaking,[10] with non-gravitational physics thus being capable of leading to fluids with negative pressure.[11] Thus, if the fluid equation of state is to be taken as a fixed, gravity independent input in the embedded gravity case as well, we would have to conclude that unless the fluid actually possesses the needed equation of state, then no (static) embedding would in fact be possible.[12]

However, in the brane-embedded case, it turns out that gravity can potentially play a different role, one in which it could be instrumental in actually fixing the fluid equation of state in the first place, with the equation of state then being output to the problem rather than input. In particular, the key difference between the embedded and non-embedded cases is that in the embedded case the matter sources are assumed to be confined to the brane, with no brane matter contribution to T_{55} being permitted. As a consequence, the G_{55} component of the empty bulk Einstein tensor has to vanish, a quite non-trivial requirement which has dynamical implications not present in a non-brane-embedded situation where the fluid is otherwise free to flow in all available spatial directions. Such a vanishing is then a constraint imposed by the geometry, and even when there are no explicit bulk matter fields to apply stresses on the brane, nonetheless there is still a non-vanishing Riemann tensor in the bulk,[13] to thus enable the bulk gravitational field to provide such stresses instead, with the bulk curvature and the brane pressure then potentially being able to dynamically adjust to each other to thereby fix the pressure on the brane and yield the brane equation of state as output.[14] Thus rather than a given input fluid equation of state imposing an output geometry on gravity, gravity instead could impose an output equation of state on the fluid.

The notion that the presence of an extra dimension might have dynamical implications and that higher dimensional gravity might play a role in stabilizing lower dimensional systems is certainly a very interesting one which requires further study. To illustrate its capability it is instructive to recall Einstein's attempt to construct a static 4-dimensional model of the universe. As is well known, in his attempt to do so Einstein introduced a cosmological constant and was then able to find a non-trivial static universe solution provided the spatial 3-curvature k of the universe was taken to be positive. In such a situation the ordinary 4-dimensional Einstein equations take the form $G^0{}_0=3k=\kappa_4^2\rho_b$, $G^i{}_j=k\delta^i_j=-\kappa_4^2 p_b\delta^i_j$ to precisely impose the self same $p_b=-\rho_b/3$ equation of state which we obtained above while fixing $k=\kappa_4^2\rho_b/3$.[15] As we now see, in the 5-dimensional $k=0$ model discussed above, the binding role played by positive k in the 4-dimensional world is instead provided by the embedding, with the constraint condition $G_{55}=0$ then providing a dynamics not otherwise present in the 4-dimensional system itself.[16]

In fact this phenomenon is actually a quite general one. Specifically if the Einstein equations are assumed to hold in the bulk, it turns out [19,20] that the induced gravitational equations on the brane actually deviate from the standard

---

[10]In such a case the harmonic phonon mode fluctuations in the lattice will still have positive pressure; it is just that they have nothing to do with the mechanism which put the atoms onto the lattice sites in the first place by minimizing the free energy. Rather they are only a perturbation around such a minimum, a minimum to which gravity however is sensitive.

[11]The $p_b=-\rho_b/3$ equation of state required above, for instance, could be associated with an isotropic network of cosmic strings, with such a negative pressure fluid potentially leading to the cosmic acceleration (see e.g. [13] for a recent review) associated with quintessence models [14].

[12]In their original papers Randall and Sundrum noted the need to have a fixed relation between bulk and brane cosmological constants in models in which $n(y)$ was initially set equal to $a(y)$. We now see that restrictions on the structure of the energy-momentum tensor are of much broader generality, in principle involving the energy densities and pressures of all brane matter sources, restrictions which are intrinsic to all embeddings (even ones into source-free bulks) and not just only to those associated with AdS_5.

[13]I.e. The very presence of the brane modifies the geometry in the source-free bulk, to thus prevent it from being the flat one that it would have been in the absence of the brane.

[14]Thus while the Israel junction condition $[n'(y=0^+)-n'(y=0^-)]/n(y=0)=\kappa_4^2(2p_b+\rho_b)/2$ associated with the embedding of a flat 2-brane in an empty 4 space would actually yield the standard weak gravity Gauss' law $[n'(y=0^+)-n'(y=0^-)]/n(y=0)=\kappa_4^2\rho_b/2$ if we were to ignore the pressure p_b on the brane (the usual weak gravity assumption), we see that the consistency of the embedding requires a very different brane pressure, one of the same order of magnitude as ρ_b, to thus yield to a weak (κ_4^2 small) gravity limit whose potential has a different normalization than that associated with a standard pressureless weak Newtonian gravity sheet. (Since the standard Newtonian potential of a sheet is just a coordinate artifact, there is no reason for it to have to correspond to the non-relativistic limit of the true gravity associated with a fully covariantized uniform sheet.)

[15]Einstein himself satisfied the relation $p_b=-\rho_b/3$ by taking $\rho_b=\rho_m+\lambda$, $p_b=-\lambda$ where ρ_m is the energy density of ordinary matter. In this solution then ordinary matter was taken to have no pressure and the cosmological constant was tuned to be given as $\lambda=\rho_m/2$. Thus already in this now quite ancient model we see the need for constraints (either input or output) on the components of the 4-dimensional matter energy-momentum tensor. In passing we note that current observations [15–17] almost a century later are apparently requiring a similar such fine-tuning between 4-dimensional matter and vacuum energy densities (for some remedies to this perplexing problem see e.g. [18]).

[16]Negative pressure solutions to the cosmic acceleration problem could thus arise as a 4-dimensional reflection of a higher dimensional embedding.

4-dimensional Einstein equations, with the additional terms that are found being explicit consequences of the embedding; viz., they do not represent new matter sources in the 4-space, but rather they arise though the constraints associated with the very existence of the embedding. In particular the authors of [19] noted that since the difference between the 4-dimensional Riemann tensor ${}^{(4)}R^{\alpha}{}_{\beta\gamma\delta}$ of a general 4-surface and the 5-dimensional Riemann tensor $R^{A}{}_{BCD}$ of some general 5-bulk into which it is embedded can be completely characterized by a function quadratic in the extrinsic curvature tensor $K_{\mu\nu}$ of the 4-surface according to the Gauss embedding formula

$$ {}^{(4)}R^{\alpha}{}_{\beta\gamma\delta}=R^{A}{}_{BCD}q_{A}{}^{\alpha}q^{B}{}_{\beta}q^{C}{}_{\gamma}q^{D}{}_{\delta}-K^{\alpha}{}_{\gamma}K_{\beta\delta}+K^{\alpha}{}_{\delta}K_{\beta\gamma}, \tag{16} $$

use of the bulk Einstein equations, the Israel junction conditions at the surface of the brane and the assumption of a $y\to -y$ symmetry around the $y=0$ brane then enable us to express the 4-dimensional Einstein tensor in terms of quantities on the brane which must necessarily be quadratic in the energy-momentum tensor of the brane. In particular, for generic metrics of the form $ds^2=dy^2+q_{\mu\nu}dx^{\mu}dx^{\nu}$ and a brane energy-momentum tensor of the specific form[17]

$$ T^{A}{}_{B}=-\Lambda_5\delta^{A}{}_{B}, \quad T_{\mu\nu}=-\lambda q_{\mu\nu}+\tau_{\mu\nu} \tag{17} $$

the authors of [19] found that the 4-dimensional Einstein tensor on the brane is given by

$$ {}^{(4)}G_{\mu\nu}=\Lambda_4 q_{\mu\nu}-8\pi G_N\tau_{\mu\nu}-\kappa_5^4\pi_{\mu\nu}-\bar{E}_{\mu\nu} \tag{18} $$

where

$$ G_N=\lambda\kappa_5^4/48\pi, \quad \Lambda_4=\kappa_5^2(\Lambda_5+\kappa_5^2\lambda^2/6)/2, $$

$$ E_{\mu\nu}=C^{A}{}_{BCD}n_A n^C q^{B}{}_{\mu}q^{D}{}_{\nu} \tag{19} $$

$$ \pi_{\mu\nu}=-\tau_{\mu\alpha}\tau_{\nu}{}^{\alpha}/4+\tau^{\alpha}{}_{\alpha}\tau_{\mu\nu}/12+q_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta}/8 -q_{\mu\nu}(\tau^{\alpha}{}_{\alpha})^2/24, \tag{20} $$

and where $\bar{E}=[E(y=0^+)+E(y=0^-)]/2$ is the mean value of $E_{\mu\nu}$ at the brane. Thus, even in the event that gravity gets to be localized to the brane, we see in general that on the brane we would expect gravity to depart from that given by just the standard 4-dimensional Einstein equations associated with a non-embedded 4-dimensional world.[18] Thus measurements within a 4-dimensional world embedded in a higher dimensional bulk would in principle be able to reveal the presence of the higher dimensional space even if the gravitational field is localized to the 4-dimensional world.[19,20]

To study further implications of such embeddings we turn now to our second soluble model, namely a static 3-brane with non-zero k embedded in an empty bulk. In the event of non-zero k the most general empty bulk solution to Eq. (8) is directly given as

$$ a^2(y)=\alpha(1-\kappa_5^2\rho_b|y|/3+k|y|^2/\alpha), \tag{21} $$

with Eq. (10) then leading to

$$ n^2(y)=(-\kappa_5^2\rho_b\alpha/3+2k|y|)^2/\alpha(1-\kappa_5^2\rho_b|y|/3+k|y|^2/\alpha), \tag{22} $$

and with Eq. (9) then entailing the equation of state

$$ p_b=-\rho_b/3-12k/\alpha\kappa_5^4\rho_b. \tag{23} $$

Thus we see that the consistency of the embedding (cf. the non-trivial vanishing of G_{55}) again imposes constraints on the brane equation of state, with the metric away from the brane now growing quadratically with distance. It is thus to

[17]This particular form was chosen so that one of the quadratic terms would then yield a term linear in $\tau_{\mu\nu}$.

[18]For a perfect fluid $\tau_{\mu\nu}=(\rho_m+p_m)U_{\mu}U_{\nu}+p_m q_{\mu\nu}$, for instance, the additional $\pi_{\mu\nu}$ tensor takes the form $\pi_{\mu\nu}=[U_{\mu}U_{\nu}(2\rho_m^2+2\rho_m p_m)+q_{\mu\nu}(\rho_m^2+2\rho_m p_m)]/12$, and thus acts like an additional perfect fluid with pressure $P=(\rho_m^2+2\rho_m p_m)/12$ and energy density $R=\rho_m^2/12$ (so that if $p_m=-\rho_m/3$, $P=+R/3$).

[19]Noting the special role played by the brane cosmological constant λ in establishing the Newton constant term in Eq. (18), we see that the effective Newton constant G_N would vary (possibly even in sign as well as magnitude) in different epochs separated by phase transitions, with early universe cosmology then potentially no longer being controlled by the Newton constant measured in a low energy Cavendish experiment. It is thus of interest to note that it is precisely an epoch dependence to both the sign and magnitude of the effective gravitational coupling constant which has recently been identified [18] as a possible solution to the cosmological constant problem.

[20]In passing, we also note a subtlety in applying Eq. (18) to the Schwarzschild problem. Specifically, even though the 4-dimensional $R_{\mu\nu}=0$ vacuum Schwarzschild solution can [21] explicitly be embedded into a 5-space with a bulk cosmological constant according to the localizing $ds^2=e^{-2|y|}[dr^2/(1-2MG/r)+r^2d\Omega-(1-2MG/r)dt^2]+dy^2$, a case where every single term in Eq. (18) is found to vanish, nonetheless, the bulk is not AdS_5 in this case. Specifically, Eq. (18) only involves the projections of the Weyl tensor along the normal direction to the brane and these components indeed do vanish in the bulk in the solution of [21]. However, explicit calculation shows that the other components of the Weyl tensor [viz. the ones not involved in Eq. (18)] do not in fact vanish (cf. $C_{0101}=2MGe^{-2|y|}/r^3$), so that (just like in our earlier discussion of the 4-dimensional Schwarzschild de Sitter metric) we find that in the presence of a Schwarzschild metric on the brane the geometry cannot be the maximally symmetric AdS_5 in the bulk (though it does become so asymptotically far from the brane). Now in their paper the authors of [19] showed that when there is a spatially inhomogeneous matter distribution on the brane, Eq. (18) then prevents the exterior bulk geometry from being pure AdS_5. Since setting $R_{\mu\nu}=0$ on the brane actually requires a delta function singularity at $r=0$ (if the $1/r$ term in g_{00} is to have a non-zero coefficient), a Schwarzschild metric on the brane actually entails an inhomogeneous source on the brane, to thus yield a non-AdS_5 bulk.

the issue of whether or not there is to be a quenching of this metric when a bulk cosmological constant is introduced to which we now turn.

III. EMBEDDING A BRANE IN A NON-EMPTY BULK

In the event of there being a bulk cosmological constant $-\rho_B=p_B=P_T=-\Lambda_5$ the structure of the solutions to the 5-dimensional Einstein equations will depend on whether there is a spatial curvature k on the brane. Thus on setting $k=0$ first and taking $a(y)$ to be a function only of $|y|$ just as before, Eq. (8) is then found to lead to

$$\frac{3}{2}\frac{d^2f(|y|)}{d|y|^2}=-\kappa_5^2\Lambda_5 f(|y|) \tag{24}$$

$$\left[3\frac{df(|y|)}{d|y|}+\kappa_5^2\rho_b f(|y|)\right]\delta(y)=0. \tag{25}$$

According to Eq. (24) (viz. a pure bulk Einstein equation which would hold even in the absence of any brane at $y=0$) and its counterpart which comes from Eq. (9), the most general allowed metric coefficient in the $k=0$ case is then found to have an unbounded exponential dependence

$$a^2(y)=f(y)=\alpha e^{\nu|y|}+\beta e^{-\nu|y|},$$

$$n(y)f^{1/2}(y)=e(y)=\alpha e^{\nu|y|}-\beta e^{-\nu|y|}, \tag{26}$$

on distance if Λ_5 is negative (viz. anti–de Sitter), with the square of the exponent ν being given by

$$\nu^2=-2\kappa_5^2\Lambda_5/3. \tag{27}$$

Thus no matter what sign we take for ν, we see that in and of itself a bulk cosmological constant does not automatically lead to exponential suppression away from the brane in this case—as we recall from Eq. (1), the AdS_5 metric is quadratic in R/z—with the most general solution to Eq. (24) in fact necessarily being unbounded, with having a $\Lambda_5\neq 0$ bulk in and of itself thus not being sufficient to guarantee brane-localization of gravity.[21] Thus we see that while the bulk might have been a pure AdS_5 bulk with only a single exponential in its metric in the absence of the brane, the introduction of the lower symmetry brane then lowers the symmetry in the bulk with the unbounded exponential no longer automatically being excluded.

In order to try to remove this undesired growing exponential anyway, we note that the insertion of Eq. (26) into the brane discontinuity formulas associated with Eq. (8) and its Eq. (9) counterpart yields

$$3\nu(\alpha-\beta)=-(\alpha+\beta)\kappa_5^2\rho_b, \tag{28}$$

$$6\nu(\alpha+\beta)=(\alpha-\beta)\kappa_5^2(\rho_b+3p_b), \tag{29}$$

relations whose solubility requires the quantity $\rho_b(\rho_b+3p_b)$ to expressly be negative. Thus if in addition to the equation of state (once again we see that embeddings entail constraints on both the bulk and brane matter fields)

$$\Lambda_5-\kappa_5^2\rho_b(\rho_b+3p_b)/12=0, \tag{30}$$

which then follows, we additionally now impose the condition

$$p_b=-\rho_b, \tag{31}$$

we will then explicitly force the coefficient β to vanish and reduce the metric to just one exponential term. Thus on dropping the β dependent term in Eq. (26) (a point we shall return to below), and retaining only the α dependent one, we then find that the discontinuity condition at the brane then fixes the sign of ν according

$$\nu=-\kappa_5^2\rho_b/3, \tag{32}$$

with the geometry now being localized to the brane according to

$$a^2(y)=\alpha e^{\nu|y|},\quad n^2(y)=\alpha e^{\nu|y|}, \tag{33}$$

when ρ_b is positive (for ρ_b negative no localization would be obtained), with Eq. (30) then yielding the compatibility condition

$$\Lambda_5+\kappa_5^2\rho_b^2/6=0. \tag{34}$$

Thus when β is set to zero the exponential dependence associated with Eq. (33) nicely quenches the linear metric dependence found in Eq. (12) in the empty bulk case just as desired, while precisely imposing on the brane the equation of state associated with a cosmological constant λ, with the condition $-\Lambda_5=\kappa_5^2\rho_b^2/6\equiv\kappa_5^2\lambda^2/6$ then entailing the vanishing of the net effective brane cosmological constant Λ_4 of the general Eq. (19),[22] just as found in the original Randall-Sundrum study.

Returning now to the more general β dependent case given in Eq. (26), we see that it is its more general equation of state $\Lambda_5-\kappa_5^2\rho_b(\rho_b+3p_b)/12=0$, rather than the restricted one of Eq. (34), which actually matches on continuously to the $p_b=-\rho_b/3$ equation of state obtained earlier as Eq. (13) in the $\Lambda_5=0$ case. The reason for this is due to the limiting process needed to extract the term linear in $|y|$ as

[21] In the original Randall-Sundrum study the brane geometry was taken to be maximally 4-symmetric (viz. Minkowski), to thus oblige the metric coefficients $a^2(y)$ and $n^2(y)$ to be equal to each other, and thereby only allow solutions with a single exponential. (Whether this single exponential itself would then be converging or diverging depends on the sign of Λ_b.) However, once the brane geometry is lowered to maximally 3-symmetric, two exponential terms with opposite sign exponents are then allowed.

[22] Equation (19) thus explains why this condition is in fact quadratic in λ.

needed for Eq. (12) from a function only containing exponentials, with it being only a linear combination of two exponentials with appropriately chosen singular coefficients $[\simeq(1\pm 1/\nu)]$ which can generate a non-vanishing linear term when the exponent $\nu=(-2\kappa_5^2\Lambda_5/3)^{1/2}$ is allowed to go to zero. Hence, by not retaining the β dependent term, we have then taken a Λ_5 dependent metric, viz. that of Eq. (33), whose $\Lambda_5\to 0$ limit does not generate any term linear in $|y|$. It is thus our assumed β vanishing $|y|=\infty$ boundary condition which takes care of the Eq. (12) empty bulk linear term, with the parameter β itself actually only vanishing when the very specific equation of state $p_b=-\rho_b$ is imposed,[23] to thus then lead to a geometry which is exponentially suppressed as we go away from the brane. As we shall now show, however, in the presence of a non-zero spatial curvature, i.e. in the presence of a non-trivial topology on the brane, even picking a brane field configuration which retains only the exponentially damped α dependent term will not in fact prove sufficient to suppress the geometry away from the brane.

In the $k\neq 0$ non-empty bulk case, Eqs. (26), (28) and (29) are found to be replaced by

$$f(y)=\alpha e^{\nu|y|}+\beta e^{-\nu|y|}-2k/\nu^2,$$

$$e(y)=\alpha e^{\nu|y|}-\beta e^{-\nu|y|}, \tag{35}$$

$$3\nu(\alpha-\beta)=-(\alpha+\beta-2k/\nu^2)\kappa_5^2\rho_b, \tag{36}$$

$$6\nu(\alpha+\beta)=(\alpha-\beta)\kappa_5^2(\rho_b+3p_b), \tag{37}$$

where again $\nu^2=-2\kappa_5^2\Lambda_5/3$. If we now drop the $\beta e^{\nu|y|}$ type term by hand by requiring the matter fields to obey $\kappa_5^2(3p_b+\rho_b)^2+24\Lambda_5=0$, the solution reduces to

$$a^2(y)=\alpha e^{\nu|y|}+3k/\kappa_5^2\Lambda_5,$$

$$n^2(y)=\alpha e^{\nu|y|}/(1+3ke^{-\nu|y|}/\alpha\kappa_5^2\Lambda_5), \tag{38}$$

where

$$\nu=(-2\kappa_5^2\Lambda_5/3)^{1/2}=-\kappa_5^2\rho_b(1+3k/\alpha\kappa_5^2\Lambda_5)/3,$$

$$p_b=-\rho_b(1+2k/\alpha\kappa_5^2\Lambda_5), \tag{39}$$

with the net brane cosmological constant Λ_4 vanishing this time if the brane matter energy density and brane cosmological constant (defined via $\rho_b=\rho_m+\lambda$) are fine-tuned according to

$$\lambda=-\rho_m(1+\alpha\kappa_5^2\Lambda_5/3k), \tag{40}$$

so that the brane matter pressure defined via $p_b=p_m-\lambda$ is then related to the brane matter energy density according to $p_m=-\rho_m/3$. As we thus see, even though the AdS_5 exponential damping factor does make an appearance in the $k\neq 0$ case, nonetheless we find that $a^2(y)$ is not asymptotically suppressed far away from the brane. Rather $a^2(y)$ tends to the non-vanishing value $3k/\kappa_5^2\Lambda_5$ if k is negative [even as $n^2(y)$ is then being suppressed], while if k is positive $n(y)$ actually becomes singular. This lack of suppression is also evidenced in the Riemann tensor, with its $R^{12}{}_{12}$ component for instance, viz.

$$R^{12}{}_{12}=f'^2/4f^2-k/f, \tag{41}$$

tending to $-\kappa_5^2\Lambda_5/3$ at large y (viz. twice the pure AdS_5 value), with the bulk embedding thus never being able to counteract the effect of the spatial curvature of the brane, no matter how judicious a choice of matter fields we may make. A similar situation is also found for the Weyl tensor where[24]

$$C^{12}{}_{12}=[-2eff''+3ef'^2-2efk-3e'ff'+2f^2e'']/12ef^2 \tag{42}$$

asymptotes to the non-vanishing value $-\kappa_5^2\Lambda_5/6$. Thus, unlike pure AdS_5, the metric associated with Eq. (38) is not conformal to flat,[25,26] with the bulk cosmological constant Λ_5 term not being able to completely quench the quadratic growth previously found for $a^2(y)$ in Eq. (21) in the $k\neq 0$, $\Lambda_5=0$ case.[27] Thus, to conclude, we see that while embedding in a higher dimensional $\Lambda_5<0$ bulk might lead to a brane-localized geometry in certain specific cases, it would appear from study of our somewhat idealized static cosmological model that such embeddings may not always lead to

[23]Since we should in principle be able to consider brane field configurations other than this particular one, we see that in general some configurations lead to suppression and some do not.

[24]For metrics of the form given in Eq. (5) all non-vanishing components of the Weyl tensor are kinematically proportional to $C^{12}{}_{12}$.

[25]As Eq. (42) also shows, even when $e(y)=f(y)=1$, $C^{12}{}_{12}$ is equal to $-k/6$ and still does not vanish. Thus even while a constant curvature 3-space embedded in an otherwise flat 4-space (viz. standard 4-dimensional Robertson-Walker) produces a 4-dimensional metric which is conformal to flat, the same is not true of the same 3-space embedded in an otherwise flat 5-space, something a bulk cosmological constant is simply unable to alter. With only the $k=0$ Robertson-Walker geometry thus being localizable by an AdS_5 embedding, it would be of interest to see whether geometries with non-zero k but with a negligibly small current era value of $\Omega_k(t)=-kc^2/\dot{R}^2(t)$ could still admit of an effective brane-localized current era gravity.

[26]It is possible to force the bulk Weyl tensor to vanish, to then force the bulk to actually be AdS_5. However, this requires the reinstatement of the divergent β type term, with the coefficients in the metric of Eq. (35) having to then obey $\alpha\beta=k^2/\nu^4\neq 0$, and with the fields then having to be related according to $\kappa_5^2\rho_b(2\rho_b+3p_b)-6\Lambda_5=0$. Thus even in this case there is still no localization of gravity. Further details of this particular case are presented separately in [22].

[27]While, as had already been noted above, the bulk geometry would not be pure AdS_5 in the event that the brane matter source was spatially inhomogeneous, we also see that even when the brane distribution is homogeneous, the bulk geometry may still not be pure AdS_5.

exponential suppression in general, with localization thus needing to be checked on a case by case basis.

Note added. As shown above, the key result of this paper is that while it is possible to localize gravity around a single Minkowski brane embedded in a non-compactified bulk with a non-zero cosmological constant, such localization is not possible for a non-flat Robertson-Walker brane embedded in the same non-compactified bulk. While beyond the specific scope and objectives of the present paper, it is nonetheless of interest to ask what would happen if the bulk fifth dimension were to be compactified into an S_1/Z_2 circular geometry with an orbifold symmetry, a two-brane setup whose cosmology was explicitly studied in [23]. Specifically, these authors considered linearized perturbations around the two Minkowski brane setup (viz. two static branes with equal and opposite cosmological constants embedded in a compactified bulk with non-zero cosmological constant) due to the addition of perfect fluid matter fields on the (explicitly assumed spatially flat) branes. In their study they found, through use of the radion stabilization mechanism (a mechanism which requires the presence of a new, additional scalar field in the bulk [24]), that the associated radion modulus fluctuation field which appears once the fifth dimension is in fact compactified then serves to stabilize the size of the fifth dimension, with standard cosmology then being recovered in the linearized limit [cf. $\lambda \gg \rho_m$, p_m in Eq. (18)]. With regard to this perturbative treatment we note that as the strengths of the brane perfect fluids are increased two things occur. First, the terms in Eq. (18) which are quadratic in ρ_m, p_m start to become important on the brane, to thus lead to departures from the standard cosmology on the brane itself. And second, as these perfect fluid matter fields build up on the brane, they start to produce new gravitational fields in the bulk. And as the study of this paper shows, in the event of non-zero spatial curvature on the brane, these ensuing bulk gravitational fields are then apparently able to non-perturbatively lead to an undoing of gravity localization altogether, and in their presence any compactification radius would therefore phenomenologically need to be microscopic.

ACKNOWLEDGMENTS

The author would like to thank Dr. A. Nayeri for introducing him to the subject of brane-localized gravity and would like to thank him and Drs. A. H. Guth, H. A. Chamblin, D. Z. Freedman and A. Karch for some very helpful discussions. The author would also like to thank Dr. R. L. Jaffe and Dr. A. H. Guth for the kind hospitality of the Center for Theoretical Physics at the Massachusetts Institute of Technology where this work was performed. This work has been supported in part by funds provided by the U.S. Department of Energy (DOE) under cooperative research agreement DF-FC02-94ER40818 and in part by grant DE-FG02-92ER40716.00.

[1] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998).
[2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436**, 257 (1998).
[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
[4] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[5] S. B. Giddings, E. Katz, and L. Randall, J. High Energy Phys. **03**, 023 (2000).
[6] C. Csaki, M. Graesser, C. Kolda, and J. Terning, Phys. Lett. B **462**, 34 (1999).
[7] P. Binetruy, C. Deffayet, and D. Langlois, Nucl. Phys. **B565**, 269 (2000).
[8] R. N. Mohapatra, A. Perez-Lorenzana, and C. A. de S. Pires, hep-ph/0003328.
[9] W. Israel, Nuovo Cimento B **44**, 1 (1966); **48**, 463(E) (1967).
[10] H. A. Chamblin and H. S. Reall, Nucl. Phys. **B562**, 133 (1999).
[11] A. Davidson and D. Karasik, Phys. Rev. D **60**, 045002 (1999).
[12] A. Vilenkin, Phys. Lett. **133B**, 177 (1983).
[13] P. D. Mannheim, Phys. Rev. D **58**, 103511 (1998).
[14] R. R. Caldwell, R. Dave, and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).
[15] A. G. Riess *et al.*, Astron. J. **116**, 1009 (1998).
[16] S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999).
[17] P. de Bernardis *et al.*, Nature (London) **404**, 955 (2000).
[18] P. D. Mannheim, astro-ph/9910093.
[19] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).
[20] M. Sasaki, T. Shiromizu, and K. Maeda, Phys. Rev. D **62**, 024008 (2000).
[21] D. Brecher and M. J. Perry, Nucl. Phys. **B566**, 151 (2000).
[22] P. D. Mannheim, Report No. MIT-CTP-2989, hep-th/0009065.
[23] C. Csaki, M. Graesser, L. Randall, and J. Terning, Phys. Rev. D **62**, 045015 (2000).
[24] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. **83**, 4922 (1999).